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On a representation of the Verhulst logistic map

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ABSTRACT

One of the simplest polynomial recursions exhibiting chaotic behavior is the logistic map $x_{n+1} = ax_n(1 - x_n)$ with $x_n, a \in \mathbb{Q} : x_n \in [0, 1]\ \forall n \in \mathbb{N}$ and $a \in (0, 4]$, the discrete-time model of the differential growth introduced by Verhulst almost two centuries ago (Verhulst, 1838) [12]. Despite the importance of this discrete map for the field of nonlinear science, explicit solutions are known only for the special cases $a = 2$ and $a = 4$. In this article, we propose a representation of the Verhulst logistic map in terms of a finite power series in the map's growth parameter a and initial value x_0 whose coefficients are given by the solution of a system of linear equations. Although the proposed representation cannot be viewed as a closed-form solution of the logistic map, it may help to reveal the sensitivity of the map on its initial value and, thus, could provide insights into the mathematical description of chaotic dynamics.

1. Introduction

Let $n \in \mathbb{N}$ and $a, x_n \in \mathbb{Q}$. The function

$$p : [0, 1] \longrightarrow \mathbb{Q} \tag{1}$$

with

$$p(x_n) = ax_n(1 - x_n) \tag{2}$$

defines a discrete recursion

$$x_{n+1} = p(x_n) \tag{3}$$

called the Verhulst logistic map [12]. It can be shown that $x_n \in [0, 1]\ \forall n$ for $a \in (0, 4]$ and $x_0 \in [0, 1]$. Moreover, as the initial value x_0 determines all future values of the system, Eq. (3) defines a deterministic Markovian system which exhibits chaotic dynamics for all $a_c < a \leq 4$ with $a_c \sim 3.569945672\ldots$ defined as the edge of chaos.

Despite its simplicity, the logistic map (3) has served since its popularization some 40 years ago (see [6]) as a prototypical dynamical system exhibiting complex chaotic behavior, and must be viewed as one of the most influential recursive equations which helped to shape the field of nonlinear science (for a recent review, see [2]). However, only two explicit closed-form solutions in the parameter space considered here are known to date, namely for the special cases $a = 2$ and $a = 4$ [10,5], and the general case can only be treated numerically or statistically (e.g., see [4]). For $a = 4$, an approach utilizing invariants of associated difference equations and their embedding into a Hilbert space by using Bose operators was explored by Steeb and Hardy [11], and can be applied to higher-dimensional maps. Previous attempts to solve (3)

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explicitly for arbitrary a include multi-dimensional functional integrals [9] and infinite-dimensional matrices [3,8], but did not provide a closed-form solution akin to those known for the aforementioned special cases. Recently, it was argued that such closed-form or smooth solutions cannot exist for generic values of a, except for a even and nonzero [13]. However, when numerically exploring the "deviation" of the generic case from the known solution at $a = 4$ as a function of a for any given n, a non-trivial yet smooth dependency can be observed [1], suggesting that at least a general solution smooth in a, albeit not necessarily closed-form, may exist.

In this article, we make use of an infinite-dimensional matrix operator acting on $\mathbb{Q}^\infty$ to describe the evolution of the logistic map from its initial state (Section 2). The explicit form of this operator is considered (Section 3), which effectively "linearizes" the discrete recursive quadratic map (3) by allowing for an explicit representation in terms of a finite power series in the map's growth parameter a and initial value x_0, with coefficients given in terms of the solution of an exponentially growing system of linear equations (Section 4). Although here also no simple closed-form solution is presented, the proposed representation might shed light on the nature of chaotic systems as well as their mathematical description.

2. Operator representation of the logistic map recursion

Lemma 1. *The logistic map (3) is equivalent to the recursive mapping*

$$A : \mathbb{Q}^\infty \longrightarrow \mathbb{Q}^\infty \tag{4}$$

with

$$\mathbf{x}_{n+1} = A \circ \mathbf{x}_n, \tag{5}$$

$n \in \mathbb{N} : n \geq 0$, *where A denotes the infinite-dimensional matrix operator*

$$a_{ij} = \begin{cases} (-1)^{j-i} \binom{i}{j-i} a^i & \forall i, j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2i \\ 0 & \textit{otherwise}, \end{cases} \tag{6}$$

$a_{ij} \in \mathbb{Q}$, *and*

$$\mathbf{x}_n = \begin{pmatrix} x_n \\ x_n^2 \\ x_n^3 \\ \vdots \end{pmatrix} \in \mathbb{Q}^\infty. \tag{7}$$

Proof. To show (6), we make use of the Carleman linearization [3,8]. To that end, consider successive powers of the logistic map (3):

$$\begin{aligned} x_{n+1} &= a x_n (1 - x_n) = a x_n - a x_n^2 \\ x_{n+1}^2 &= a^2 x_n^2 (1 - x_n)^2 = a^2 x_n^2 - 2a^2 x_n^3 + a^2 x_n^4 \\ &\vdots \\ x_{n+1}^m &= a^m x_n^m (1 - x_n)^m = a^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} x_n^{k+m} \\ &\vdots \end{aligned}$$

Defining the vector $\mathbf{x}_n$ according to (7), this system of nonlinear equations can be put into the form

$$\mathbf{x}_{n+1,i} = \sum_{j=1}^{\infty} a_{ij}\, \mathbf{x}_{n,j}, \tag{8}$$

where $\mathbf{x}_{n,i}$ denotes the ith component of $\mathbf{x}_n$ and a_{ij} the components of the matrix

$$A = \begin{pmatrix} a & -a & 0 & 0 & 0 & 0 & \\ 0 & a^2 & -2a^2 & a^2 & 0 & 0 & \\ 0 & 0 & a^3 & -3a^3 & 3a^3 & -a^3 & \cdots \\ 0 & 0 & 0 & a^4 & -4a^4 & 6a^4 & \\ 0 & 0 & 0 & 0 & a^5 & -5a^5 & \\ & & & \vdots & & & \end{pmatrix},$$

that is

$$
\begin{aligned}
a_{ij} &= \sum_{k=0}^{i} \delta_{i,j-k} (-1)^k \binom{i}{k} a^i \\
&\equiv \begin{cases} (-1)^{j-i} \binom{i}{j-i} a^i & \forall i,j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2i \\ 0 & \text{otherwise.} \quad \square \end{cases}
\end{aligned}
$$

The operator A is an infinite-dimensional strict upper triangular matrix with, for $a \neq 0$, non-vanishing entries for $i \leq j \leq 2i$ (wedge-matrix) and non-zero diagonal elements, hence invertible and diagonalizable. Moreover, for any given i and finite n, the sum in (8) will always terminate for $j > 2i$.

With Lemma 1, we can now formulate

Proposition 1. *The logistic map* (3) *takes the explicit form*

$$\mathbf{x}_n = A^n \circ \mathbf{x}_0, \tag{9}$$

$n \in \mathbb{N} : n \geq 1$, *where* A^n *denotes the* nth *power of the matrix operator* A, *i.e.*

$$
(a^n)_{ij} = \begin{cases} (-1)^{j-i} a^i \binom{i}{j-i} & \text{for } n = 1 \\ (-1)^{j-i} a^i \sum_{l_1=i}^{2i} \sum_{l_2=l_1}^{2l_1} \cdots \sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}} a^{\sum_{p=1}^{n-1} l_p} \binom{i}{l_1 - i} \binom{l_{n-1}}{j - l_{n-1}} \prod_{p=1}^{n-2} \binom{l_p}{l_{p+1} - l_p} & \text{for } n \geq 2 \end{cases} \tag{10}
$$

$\forall i, j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2^n i$. *All other* $(a^n)_{ij}$ *vanish.*

Proof. Repeated application of the recursive map (5) yields

$$\mathbf{x}_n = \underbrace{A \circ A \circ \cdots \circ A}_{n \text{ times}} \circ \mathbf{x}_0.$$

As A is a matrix, these n successive applications of the operator A are equivalent to taking the nth power of A, thus yielding (9).

To show (10), one directly evaluates A^n. For $n = 1$, the matrix operator (6) itself is obtained. For $n \geq 2$, one has

$$
\begin{aligned}
(a^n)_{ij} &= \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \cdots \sum_{l_{n-1}=1}^{\infty} a_{il_1} a_{l_1 l_2} \cdots a_{l_{n-1} j} \\
&= \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \cdots \sum_{l_{n-1}=1}^{\infty} a_{il_1} a_{l_{n-1} j} \prod_{p=1}^{n-2} a_{l_p l_{p+1}} \\
&= \sum_{l_1=i}^{2i} \sum_{l_2=l_1}^{2l_1} \cdots \sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}} (-1)^{l_1 - i + j - l_{n-1}} \prod_{p=1}^{n-2} \left\{ (-1)^{l_{p+1} - l_p} \right\} a^{i + l_{n-1} + \sum_{p=1}^{n-2} l_p} \binom{i}{l_1 - i} \binom{l_{n-1}}{j - l_{n-1}} \prod_{p=1}^{n-2} \binom{l_p}{l_{p+1} - l_p}.
\end{aligned}
$$

In the last step, the explicit form of a_{ij} was used. As successive terms in the first product will cancel, the last equation can be simplified, yielding (10). Moreover, due to the wedge-shape of a_{ij}, this expression holds $\forall i, j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2^n i$, with all other $(a^n)_{ij}$ being zero. $\square$

We note that Proposition 1 provides the explicit form of the original recursive map (5) in terms of a finite power series in the parameter a, with minimum order ni and maximum order $i + \sum_{p=1}^{n-1} 2^p i = (2^n - 1)i$ for any given $i \geq 1$ and $n \geq 1$. However, the number of terms in this power series grows exponentially with n, and a closed-form expression is made difficult due to the presence of $(n-1)$ nested sums over products of binomial coefficients.

3. Finite power series expansion of the operator representation

Proposition 1 can be used to represent the logistic map explicitly in terms of a finite power expansion in both the parameter a and initial value x_0. To that end, we introduce the following set of coefficients $V_{kij}^{(n)} \in \mathbb{N}$:

Definition 1.

$$V_{kij}^{(n)} := \underbrace{\sum_{l_1=i}^{2i}\sum_{l_2=l_1}^{2l_1}\cdots\sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}}}_{l_1+l_2+\cdots+l_{n-1}=k} \binom{i}{l_1-i}\binom{l_{n-1}}{j-l_{n-1}}\prod_{p=1}^{n-2}\binom{l_p}{l_{p+1}-l_p} \tag{11}$$

$\forall n \in \mathbb{N} : n \geq 2,\ \forall i, j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2^n i$ and $k \in \mathbb{N} : (n-1)i \leq k \leq (2^n-2)i$. All other $V_{kij}^{(n)}$ are zero.

The coefficients defined as such no longer depend on the parameter a, and thus yield

$$(a^n)_{ij} = (-1)^{j-i} a^i \sum_{k=(n-1)i}^{(2^n-2)i} V_{kij}^{(n)}\, a^k \tag{12}$$

for the nth power ($n \geq 2$) of the matrix operator A. Interestingly, the coefficients $V_{kij}^{(n)}$ link the dynamics of the logistic map to a particular partitioning of integers, specifically the subset of partitions of a given integer k into sums of $(n-1)$ integers $l_p, p \in [1, n-1]$ where $i \leq l_1 \leq 2i$ and $l_{p-1} \leq l_p \leq 2l_{p-1}$ for $p > 1$. As we will see below, the logistic map can be completely formulated in terms of $V_{kij}^{(n)}$.

To further simplify notation, we also define a set of functions $V_{ij}^{(n)}(a) \in \mathbb{Q}$ according to

Definition 2.

$$\begin{aligned} V_{ij}^{(n)}(a) &:= \sum_{k=(n-1)i}^{(2^n-2)i} V_{kij}^{(n)}\, a^k \\ &= \sum_{l_1=i}^{2i}\sum_{l_2=l_1}^{2l_1}\cdots\sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}} a^{\sum\limits_{p=1}^{n-1} l_p} \binom{i}{l_1-i}\binom{l_{n-1}}{j-l_{n-1}}\prod_{p=1}^{n-2}\binom{l_p}{l_{p+1}-l_p} \end{aligned} \tag{13}$$

$\forall n \in \mathbb{N} : n \geq 2$ and $\forall i, j \in \mathbb{N} : i \geq 1,\ i \leq j \leq 2^n i$. All other $V_{ij}^{(n)}(a)$ are identically zero.

Functions defined as such depend explicitly on a, and yield

$$(a^n)_{ij} = (-1)^{j-i}\, V_{ij}^{(n)}(a)\, a^i \tag{14}$$

for the nth power ($n \geq 2$) of the matrix operator A.

Lemma 2. *The functions $V_{ij}^{(n)}(a)$ obey the recursive algebraic relations*

$$V_{ij}^{(n+1)}(a) = \sum_{q=0}^{i} a^{i+q}\binom{i}{q} V_{i+q,j}^{(n)}(a) \tag{15}$$

$$V_{ij}^{(n+1)}(a) = \sum_{q=i}^{2^n i} a^{q}\binom{q}{j-q} V_{iq}^{(n)}(a) \tag{16}$$

$\forall n \in \mathbb{N} : n \geq 2$ *and* $\forall i, j \in \mathbb{N} : i \geq 1, i \leq j \leq 2^{n+1} i$.

Proof. To show (15), we use the definition of the functions (13) for $(n+1)$ and sum over l_1. After changing the summation variable to $q = l_1 - i$, one obtains

$$\begin{aligned} V_{ij}^{(n+1)}(a) &= \sum_{q=0}^{i}\left\{\sum_{l_2=i+q}^{2(i+q)}\cdots\sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}}\sum_{l_n=l_{n-1}}^{2l_{n-1}} a^{i+q+\sum_{p=2}^{n} l_p}\binom{i}{q}\binom{l_n}{j-l_n}\prod_{p=1}^{n-1}\binom{l_p}{l_{p+1}-l_p}\right\} \\ &= \sum_{q=0}^{i} a^{i+q}\binom{i}{q}\left\{\sum_{l_1=i+q}^{2(i+q)}\cdots\sum_{l_{n-2}=l_{n-3}}^{2l_{n-3}}\sum_{l_{n-1}=l_{n-2}}^{2l_{n-2}} a^{\sum_{p=1}^{n-1} l_p}\binom{i+q}{l_1-(i+q)}\binom{l_{n-1}}{j-l_{n-1}}\prod_{p=1}^{n-2}\binom{l_p}{l_{p+1}-l_p}\right\} \\ &= \sum_{q=0}^{i} a^{i+q}\binom{i}{q} V_{i+q,j}^{(n)}. \end{aligned}$$

Here we relabeled the summation variables according to $l_{p+1} \to l_p$ and made use of the definition of $V_{ij}^{(n)}$.

To show (16), we first argue that the $(n-1)$ nested sums in (13) can be decoupled by changing the summation limits for each l_p. Through simple inspection, one infers that the minimal value each l_p can take is i, and the maximal value cannot exceed $2^p i$. With this, the l_p-relevant term in the argument of (13) is given by $\sum_{l_p=l_{p-1}}^{2l_{p-1}} \binom{l_{p-1}}{l_p-l_{p-1}} \binom{l_p}{l_{p+1}-l_p}$ with $i \le l_{p-1} \le 2^{p-1} i$. Given a l_{p-1}, the argument of this term will vanish if $l_p < l_{p-1}$, leaving the first non-vanishing term for $l_p = l_{p-1}$ and all other terms with $i \le l_p < l_{p-1}$ zero. Similarly, given a l_{p-1}, the argument will vanish for $l_p > 2l_{p-1}$ due to $\binom{l_{p-1}}{l_p-l_{p-1}} = \binom{l_{p-1}}{2l_{p-1}-l_p}$, leaving the last non-vanishing term for $l_p = 2l_{p-1}$ and all other terms with $2l_{p-1} < l_p \le 2 \cdot 2^{p-1} i$ zero. With this, (13) takes for $(n+1)$ the form

$$V_{ij}^{(n+1)}(a) = \sum_{l_1=i}^{2i} \sum_{l_2=i}^{4i} \cdots \sum_{l_{n-1}=i}^{2^{n-1}i} \sum_{l_n=i}^{2^n i} a^{\sum_{p=1}^{n} l_p} \binom{i}{l_1-i} \binom{l_n}{j-l_n} \prod_{p=1}^{n-1} \binom{l_p}{l_{p+1}-l_p}.$$

Performing the sum over l_n yields

$$V_{ij}^{(n+1)}(a) = \sum_{q=i}^{2^n i} a^q \binom{q}{j-q} \left\{ \sum_{l_1=i}^{2i} \sum_{l_2=i}^{4i} \cdots \sum_{l_{n-1}=i}^{2^{n-1}i} a^{\sum_{p=1}^{n-1} l_p} \binom{i}{l_1-i} \binom{l_{n-1}}{q-l_{n-1}} \prod_{p=1}^{n-2} \binom{l_p}{l_{p+1}-l_p} \right\},$$

where we relabeled $l_n \to q$. The term in the curly brackets is identical to $V_{ij}^{(n)}(a)$ for $j \to q$, thus proving (16). □

By utilizing the definition of the functions $V_{ij}^{(n)}(a)$ in terms of $V_{kij}^{(n)}$ given in Eq. (13), corresponding recursive relations between the coefficients can be found.

Lemma 3. *The coefficients $V_{kij}^{(n)}$ obey the recursive algebraic relations*

$$V_{kij}^{(n+1)} = \underbrace{\sum_{p=i}^{2i} \sum_{q=(n-1)p}^{(2^n-2)p}}_{p+q=k} \binom{i}{p-i} V_{qpj}^{(n)} \tag{17}$$

$$V_{kij}^{(n+1)} = \underbrace{\sum_{p=ni}^{(2^n-1)i} \sum_{q=0}^{(2^n-1)i}}_{p+q=k} \binom{q+i}{j-q-i} V_{p-i,i,q+i}^{(n)} \tag{18}$$

$\forall n \in \mathbb{N} : n \ge 2$, $\forall i, j \in \mathbb{N} : i \ge 1$, $i \le j \le 2^{n+1} i$ and $k \in \mathbb{N} : ni \le k \le (2^{n+1}-2)i$.

Proof. Both relations can be shown by inserting (13) on both sides of the relations given in Lemma 2, reordering the sums with respect to powers of a, and comparing coefficients of the resulting finite power series in a.

Specifically, inserting (13) into (15) yields

$$\begin{aligned}
\sum_{k=ni}^{(2^{n+1}-2)i} a^k V_{kij}^{(n+1)} &= \sum_{q=i}^{2i} \sum_{k=(n-1)q}^{(2^n-2)q} a^{q+k} \binom{i}{q-i} V_{kqj}^{(n)} \\
&= \sum_{l=ni}^{(2^{n+1}-2)i} a^l \left\{ \underbrace{\sum_{q=i}^{2i} \sum_{k=(n-1)q}^{(2^n-2)q}}_{q+k=l} \binom{i}{q-i} V_{kqj}^{(n)} \right\},
\end{aligned}$$

where in the last step we collected on the right-hand side all terms proportional to a^l, $l \in [ni, (2^{n+1}-2)i]$. Comparing the coefficients for terms proportional to a given power of a on both sides yields, after renaming the summation variables, relation (17).

The correctness of relation (18) can be shown in a similar fashion. □

We note that relation (17) links all coefficients $V_{kij}^{(n+1)}$ at step $(n+1)$ to a sum over coefficients $V_{qpj}^{(n)}$ with $q \in [(n-1)i, (2^n-2)i]$, $p \in [i, 2i]$ and $p+q=k$ at step n. Equivalently, relation (18) is a recursive equation in n which links all coefficients $V_{kij}^{(n+1)}$ at step $(n+1)$ to a sum over coefficients $V_{piq}^{(n)}$ with $p \in [(n-1)i, (2^n-2)i]$, $q \in [i, 2i]$ and $p+q=k$ at step n.

Introducing for simplicity of notation

$$\mathcal{V}_{kj}^{(n)} := (-1)^{j-1} V_{k1j}^{(n)}, \tag{19}$$

we can now formulate

Proposition 2. *The logistic map (3) is equivalent to the explicit finite power series*

$$x_n = a \sum_{j=1}^{2^n} \sum_{k=n-1}^{2^n-2} \mathcal{V}_{kj}^{(n)} a^k x_0^j \tag{20}$$

for $n \in \mathbb{N} : n \geq 2$, with the coefficients defined recursively in n through

$$\mathcal{V}_{kj}^{(n+1)} = \sum_{q=1}^{2^n} (-1)^{j-q} \binom{q}{j-q} \mathcal{V}_{k-q,q}^{(n)} \tag{21}$$

for all $k \in [n, 2^{(n+1)} - 2]$ and $j \in [1, 2^{n+1}]$ with the initial values

$$\begin{aligned}
&\mathcal{V}_{11}^{(2)} = \binom{1}{0}\binom{1}{0} = 1, && \mathcal{V}_{21}^{(2)} = \binom{1}{1}\binom{2}{-1} = 0, \\
&\mathcal{V}_{12}^{(2)} = -\binom{1}{0}\binom{1}{1} = -1, && \mathcal{V}_{22}^{(2)} = -\binom{1}{1}\binom{2}{0} = -1, \\
&\mathcal{V}_{13}^{(2)} = \binom{1}{0}\binom{1}{2} = 0, && \mathcal{V}_{23}^{(2)} = \binom{1}{1}\binom{2}{1} = 2, \\
&\mathcal{V}_{14}^{(2)} = -\binom{1}{0}\binom{1}{3} = 0, && \mathcal{V}_{24}^{(2)} = -\binom{1}{1}\binom{2}{2} = -1.
\end{aligned}$$

Proof. Using the operator form (Proposition 1), we first observe that the first row in (9) yields the expression linear in x_n, thus

$$\begin{aligned}
x_n &= \sum_{j=1}^{\infty} (a^n)_{1j} x_0^j \\
&= \sum_{j=1}^{2^n} (-1)^{j-1} a \sum_{k=n-1}^{2^n-2} V_{k1j}^{(n)} a^k x_0^j .
\end{aligned}$$

Here, we made use of the upper-triangular wedge-like structure of the operator matrix A and its powers in order to truncate the summation over j. Inserting the definition of $\mathcal{V}_{kj}^{(n)}$, Eq. (19), yields (20).

The recursive form of the coefficients $\mathcal{V}_{kj}^{(n)}$ can be obtained from (18) for $i = 1$ and using (19). Finally, the initial values are deduced from definition (11) using $i = 1$. □

Due to their original definition as sums over products of binomial coefficients, Eqs. (11) and (19), the coefficients $\mathcal{V}_{kj}^{(n)}$ are integers with rapidly growing absolute value for increasing n. Moreover, the number of these coefficients for a given n is exponentially growing with n, but the recurrence (21) is sufficient to calculate all $2^{n+1}(2^{n+1} - n - 1)$ coefficients $\mathcal{V}_{kj}^{(n+1)}$ from the $2^n(2^n - n)$ coefficients at step n. To illustrate both points, we list in Table 1 all non-zero $\mathcal{V}_{kj}^{(n)}$ up to $n = 4$ and in Table 2 all coefficients for $n = 5$.

4. "Linearized" representation of the logistic map

Although the representation of the logistic map in Proposition 2 is explicit in terms of a finite power-series in a and x_0, the coefficients are given in form of a linear recursive relation with an exponentially growing number of terms for increasing n. As the number of terms in this recursion depends on the step n, classical methods, such as the generating function approach [14], cannot be employed to obtain an explicit closed-form expression for $\mathcal{V}_{kj}^{(n)}$. However, using the well-known non-trivial fixed-points of the original system

$$x_a = \frac{a-1}{a} \tag{22}$$

for any given $a \in (0, 4]$, we can represent the recursion (21) in terms of a system of linear equations. Relabeling $\mathcal{V}_{kj}^{(n)} \longrightarrow \mathcal{V}_q^{(n)}$ with $q = (j-1)(2^n - n) + k - n + 2,\ q \in [1, 2^n(2^n - n)]$, we have

Proposition 3 (*"Linearized" Representation*)**.** *The logistic map (3) is equivalent to the explicit finite power series*

$$x_n = a \sum_{j=1}^{2^n} \sum_{k=n-1}^{2^n-2} \mathcal{V}_{kj}^{(n)} a^k x_0^j \tag{23}$$

Table 1
Coefficients $\mathcal{V}_{kj}^{(n)}$ for $n = \{2, 3, 4\}$. The gray boxes indicated the coefficients used to calculate $\mathcal{V}_{22,14}^{(5)}$ as an illustrative example of the recursive relation (21) (see Table 2).

k	j															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$n = 2$																
1	1	−1														
2		−1	2	−1												
$n = 3$																
2	1	−1														
3		−1	2	−1												
4		−1	2	−1												
5			2	−6	6	−2										
6				−1	4	−6	4	−1								
$n = 4$																
3	1	−1														
4		−1	2	−1												
5		−1	2	−1												
6		−1	4	−7	6	−2										
7			2	−7	10	−8	4	−1								
8			2	−7	10	−8	4	−1								
9				−6	24	−36	24	−6								
10				−1	10	−36	64	−61	30	−6						
11					4	−22	52	−70	60	−34	12	−2				
12						−6	36	−90	120	−90	36	−6				
13							4	−28	84	−140	140	−84	28	−4		
14								−1	8	−28	56	−70	56	−28	8	−1

for $n \in \mathbb{N} : n \geq 2$, with the coefficients defined as the solution to the system of linear equations given by

$$C_p^{(n)} = D_{pq}^{(n)} \mathcal{V}_q^{(n)}, \tag{24}$$

where

$$\begin{aligned} D_{pq}^{(n)} &= a_p^{n+(q-1) \bmod (2^n-n)} \left(\frac{a_p - 1}{a_p} \right)^{\frac{1}{2^n-n}(q-1-(q-1) \bmod (2^n-n))+1} \\ &= a_p^{n+q-1-(2^n-n)\left\lfloor \frac{q-1}{2^n-n} \right\rfloor} \left(\frac{a_p - 1}{a_p} \right)^{\left\lfloor \frac{q-1}{2^n-n} \right\rfloor + 1} \end{aligned} \tag{25}$$

and

$$C_p^{(n)} = \frac{a_p - 1}{a_p}$$

for $2^n(2^n - n)$ different non-trivial fixed-points $\{a_p \in \mathbb{Q},\ a_p \in (0, 4];\ p \in [1, 2^n(2^n - n)]\}$ of the logistic map.

Proof. Eqs. (24) and (25) follow straightforward by successively inserting fixed-points (22) for a chosen a_p into the left-hand and right-hand side of (23). □

Proposition 3 provides a fully linearized representation of the Verhulst logistic map on the expense of the size of the associated system of linear equations. However, although, in principle, (24) can be explicitly solved, it is of little use practically, especially for larger n.

5. Conclusion

In this paper we have proposed a "linearized" representation of the Verhulst logistic map, a second order recursive relation exhibiting both periodic and chaotic behavior depending on its parameter a. To that end, we first made use of the Carleman linearization and expresses the logistic map explicitly in terms of a matrix operator acting on an infinite-dimensional $\mathbb{Q}$-valued vector space (Proposition 1). The evolution of the logistic map is here given through successive powers of this matrix operator acting upon an initial state vector $\mathbf{x}_0$.

Next, by using the explicit form of this operator, we expressed the logistic map explicitly in terms of a finite power series in the initial state value x_0 and the map's parameter a (Proposition 2). Although the obtained expansion cannot be viewed as

Table 2

Coefficients $\mathcal{V}_{kj}^{(n)}$ for $n=5$. To illustrate the recursive relation (21), the gray box indicates the result of the combination of $\mathcal{V}_{kj}^{(4)}$ (see Table 1, gray boxes) in order to obtain $\mathcal{V}_{22,14}^{(5)}$: $\mathcal{V}_{22,14}^{(5)} = \binom{8}{6}\mathcal{V}_{14,8}^{(4)} - \binom{9}{5}\mathcal{V}_{13,9}^{(4)} + \binom{10}{4}\mathcal{V}_{12,10}^{(4)} - \binom{11}{3}\mathcal{V}_{11,11}^{(4)}$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$n=5$																
4	1	−1														
5		−1	2	−1												
6		−1	2	−1												
7		−1	4	−7	6	−2										
8		−1	4	−8	10	−8	4	−1								
9			4	−13	16	−10	4	−1								
10			2	−13	34	−44	28	−7								
11			2	−13	40	−74	88	−67	30	−6						
12				−7	38	−94	140	−137	90	−40	12	−2				
13				−6	34	−94	172	−226	210	−130	48	−8				
14				−1	28	−134	296	−389	364	−284	188	−92	28	−4		
15					10	−86	320	−669	862	−718	412	−190	84	−32	8	−1
16					4	−56	280	−749	1252	−1412	1112	−610	224	−52	8	−1
17						−22	196	−784	1832	−2738	2708	−1786	784	−232	48	−6
18						−6	88	−515	1700	−3618	5272	−5368	3 780	−1760	488	−61
19							36	−322	1346	−3490	6260	−8176	7 952	−5776	3 080	−1150
20							4	−118	864	−3206	7400	−11 694	13 348	−11 344	7 272	−3510
21								−28	344	−1898	6228	−13 570	20 768	−23 044	18 872	−11 488
22								−1	92	−874	3992	−11 308	22 100	−31 492	33 704	−27 469
23									8	−212	1724	−7370	19 812	−36 480	48 272	−47 310
24										−28	420	−2806	11 132	−29 376	54 576	−73 530
25											56	−700	4 088	−14 784	36 960	−67 452
26												−70	868	−4984	17 584	−42 658
27													56	−732	4 424	−16 380
28														−28	392	−2548
29															8	−120
30																−1

k \ j	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$n=5$																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																
16																
17																
18																
19	270	−30														
20	1260	−330	60	−6												
21	5160	−1650	340	−34												
22	17 100	−8094	2 880	−750	132	−12										
23	35 088	−20 036	8 924	−3110	836	−168	24	−2								
24	72 792	−53 004	28 132	−10 698	2 860	−536	72	−6								
25	92 400	−96 096	75 768	−44 660	19 096	−5600	1 008	−84								
26	75 460	−100 716	103 488	−82 698	51 436	−24 640	8 848	−2254	364	−28						
27	41 496	−76 076	104 104	−108 108	85 800	−52 052	24 024	−8372	2 184	−420	56	−4				
28	10 192	−28 028	56 056	−84 084	96 096	−84 084	56 056	−28 028	10 192	−2548	392	−28				
29	840	−3640	10 920	−24 024	40 040	−51 480	51 480	−40 040	24 024	−10 920	3640	−840	120	−8		
30	16	−120	560	−1820	4 368	−8008	11 440	−12 870	11 440	−8008	4368	−1820	560	−120	16	−1

a closed-form solution for generic a, it provides a finite representation, smooth in both the parameter a and initial value x_0. The coefficients of this series $\mathcal{V}_{kj}^{(n)}$, defined in (19) together with (11), are $\mathbb{Z}$-valued numbers with rapidly growing absolute values involving a subset of partitions of natural numbers and obeying a set of recursive algebraic relations (Lemma 3).

Although this power series expansion is of little practical use for numerical calculations due to the exponentially growing number of coefficients with increasing step n, it provides an insight into the nature of the original chaotic recursion. In particular, the order of the power series grows exponentially with step n, thus demonstrating explicitly the sensitivity of the system to both its parameter a as well as initial condition x_0, the defining characteristic of chaotic systems. Moreover, for any given order in x_0 and a in the power series, the coefficients indirectly depend on n. This effectively leads to a "mixing" of contribution of the various orders in the power expansion for successive steps, as illustrated in Tables 1 and 2 (gray boxes).

The final representation (Proposition 3) makes use of the fixed-points of the logistic map, leading to a formal representation of the coefficients $\mathcal{V}_{kj}^{(n)}$ in terms of solutions of a system of linear equations (24). Although, in principle, a solution to this system can be found, it is of little or no practical interest due to the size of the system. However, this representation can be viewed as an effective "linearization" of the chaotic system in question, a linearization achieved at the expense of an exponentially growing size of the linear system in $\mathcal{V}_{kj}^{(n)}$.

Although the proposed representation can be viewed as an explicit form of the Verhulst logistic map, the prospects for numerical application are challenging. Numerical evaluation of this representation will necessarily involve either calculating recursively an exponentially growing number of coefficients, calculating the nth power of a infinite-dimensional matrix, or solving an exponentially growing system of linear equations. However, while the "classical" double-precision implementation of the logistic map can be viewed with algorithmic complexity $O(n)$ in time and $O(1)$ in memory, we note that in order to avoid round-off errors in any practical implementation [7,15], arbitrary precision methods must be employed, thus implicating exponential complexity into the problem. The representation presented here has the full precision of an analytic expression, hence allowing to evaluate the logistic map, in principle, to arbitrary precision. Moreover, we hope that this representation sheds some light on the nature of chaotic systems, and potentially paves the way for a discrete mathematics of large numbers which might be more suitable for describing nonlinear or even chaotic systems.

We note, finally, that the proposed representation of the Verhulst logistic map is applicable to general polynomial recursions, thus potentially allowing for a formulation of such maps within a unified mathematical framework.

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